Implementing the *Australian Curriculum for Mathematics K to 10*: The place of problem solving and reasoning in mathematics learning.

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The new *Australian Curriculum for Mathematics K to 10* is represented as three content strands and four proficiency strands (Understanding, Fluency, Problem solving and Reasoning). Teachers tend to focus on the content strands since these are more likely to be assessed, particularly in 'high stakes' testing. However, the proficiency strands support deep learning and rich understandings, particularly if they are implemented in ways supported by advice from the research. Incorporating investigations into mathematics lessons provides opportunities for students to experience problem-solving processes and reasoning. Strategies supporting such approaches include questioning based on Bloom’s Taxonomy as well as open-ended tasks. To cater for the range of student knowledge, skills and understandings in mathematics classrooms, building an investigation from simple questions to more complex multi-step questions provides appropriate scaffolding and support.

The draft Australian Curriculum for Mathematics from Kindergarten to Year 10 (ACARA, 2010) presents the mathematics content in three strands – Number and algebra; Measurement and geometry; and Statistics and probability. There are also four proficiency strands – Understanding, Fluency, Problem solving and Reasoning, which describe the ‘applications or actions’ of mathematics. These proficiencies replace the more familiar Working Mathematically strand from the NSW curriculum (BOSNSW, 2003).

Teachers have had many opportunities to build knowledge about teaching problem solving and using problems as a focus of learning in mathematics (Cai, 2003). In Australia advice to teachers has been provided in a range of publications including books (e.g., Lovitt & Clarke, 1988) and professional journals, in national curriculum statements (e.g., Australian Education Council, 1991) as well as in state and territory curriculum documents (e.g. BOSNSW, 2003). Such advice has been accompanied by pre-service and in-service programs to change teaching practices from more traditional approaches to contemporary or reform methods where teachers use non-routine problems and problem-centred tasks (Anderson & Bobis, 2005). Given the amount of policy advice and resource development, research in classrooms suggests there have been limited opportunities for Australian students to solve problems other than those of low procedural complexity (Stacey, 2003). It is possible that one of the main constraints on implementation is the type of questions included in assessment tasks (particularly high-stakes assessment) and in resources such as school textbooks (Doorman et al., 2007; Kaur & Yeap, 2009; Vincent & Stacey, 2008) as well as the types of tasks typically used by teachers.

For problem solving and reasoning to become regular components of mathematics learning in classrooms throughout Australia we need to provide explicit information about the meanings of ‘problem solving’ and ‘reasoning’ as well as examples of rich learning and assessment tasks that provide opportunities for students to demonstrate these proficiencies. In addition, teachers require advice about the development of the full range of problem-solving skills and dispositions as described by Stacey (2005). While there will be challenges for teachers to begin to implement the new Australian curriculum for mathematics, one of the biggest challenges will be to embed the proficiencies into the content strands within programs, within teaching and learning experiences in classrooms, and within all assessment of mathematics. This paper provides suggestions and advice to support the implementation of problem solving and reasoning in mathematics classrooms.

*Problem Solving and Reasoning in the Mathematics Curriculum*

Problem solving is recognised as an important life skill involving a range of processes including analysing, interpreting, reasoning, predicting, evaluating and reflecting. It is either an overarching goal or a fundamental component of the school mathematics curriculum in many countries. In NSW, problem solving was made explicit in the mathematics syllabuses developed and introduced into primary and junior secondary classrooms in the Eighties. Advice was provided about how problem solving could be implemented including the possibilities associated with teaching for problem solving, teaching about problem solving and teaching through problem solving. Teaching for problem solving involves teaching the mathematical content first for later use in solving problems. Teaching about problem solving involves teaching strategies (heuristics) and processes to improve problem-solving competence. Teaching through problem solving requires teaching mathematical content by first presenting a non-routine problem to students so that the need to learn the content becomes apparent.

While the advice and recommendations in these early syllabus documents was useful, problem solving was still presented as a separate strand or integrated as teaching ideas and not necessarily visible to teachers. The more recent NSW syllabuses for mathematics from Kindergarten to Year 10 adopted Working Mathematically as the process strand with five components – Questioning, Applying strategies, Communicating, Reasoning and Reflecting. I argue later in this paper that these five processes can be used as components of the problem-s solving cycle but evidence suggests this approach has had limited implementation (Cavanagh, 2006).

As noted in the Shape of the Australian Curriculum: Mathematics (NCB 2009, p. 5) document, the term Working Mathematically was not considered to adequately represent the full range of actions so the new proficiencies have been adapted from Kilpatrick, Swafford and Findell (2001). From the Draft curriculum document (ACARA, 2010), the descriptions of each of the proficiency strands are presented in Table 1.

Table 1 – The proficiency strands and descriptions from the Draft Australian Curriculum: Mathematics K to 10 (ACARA, 2010, p. 3)

<table>
<thead>
<tr>
<th>Proficiency</th>
<th>Description</th>
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<tbody>
<tr>
<td>Understanding</td>
<td>Students build robust knowledge of adaptable and transferable mathematical concepts, make connections between related concepts and develop the confidence to use the familiar to develop new ideas, and the ‘why’ as well as the ‘how’ of mathematics.</td>
</tr>
<tr>
<td>Fluency</td>
<td>Students develop skills in choosing appropriate procedures, carrying out procedures flexibly, accurately, efficiently and appropriately, and recalling factual knowledge and concepts readily.</td>
</tr>
<tr>
<td>Problem Solving</td>
<td>Students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively.</td>
</tr>
<tr>
<td>Reasoning</td>
<td>Students develop increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying, and generalising.</td>
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</table>

Kilpatrick et al. (2001, p. 5) included a fifth proficiency, productive disposition, described as “habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy”. It is disappointing that this proficiency was not included in the Draft curriculum document since it is critical that we focus on developing positive dispositions in mathematics, particularly given recent reports highlighting negative views about both the content and teaching of mathematics in the early secondary years (McPhan, et al., 2008). While separate descriptions are provided for each proficiency, it is important to note that the five proficiencies are “interwoven and interdependent” (Kilpatrick et al., 2001, p. 5). The question remains: will these new proficiencies encourage teachers to focus not just on mathematics content but also on these important processes?

The description for problem solving from Table 1 suggests students need to actively engage with a range of important processes during mathematics lessons. For this to occur teachers will need to select tasks, which allow for student choice about the mathematics they might use and the problem-solving strategies they select to model and investigate their solution. According to the NCTM Standards (2000, p. 52) “problem solving means engaging in a task for which the solution method is not known in advance”. So problem solving frequently involves investigating new and somewhat challenging situations that require time and effort. For many students problem solving needs to be more than just doing questions which are applications of the mathematics they are learning right now. So what types of questions and tasks are suitable?

Question Types

Several useful resources are available, which provide advice to teachers about extending their repertoire of question types so that students engage in problem solving and reasoning. Bills, Bills, Watson and Mason (2004) list a range of question types, which promote discussion and deeper thinking. One of my favourites is ‘always/sometimes/never’. Students are provided with a statement for which they need to consider whether it is always true, sometimes true or never true. Examples include:

- multiplication makes bigger;
- parallelograms have no axes of symmetry;
- when you add two numbers, you get the same result as when you multiply them;
- \((a + b)^2 = a^2 + b^2\); and
- all circles are similar.
Presenting their solutions with counter examples and generalisations provides evidence of students’ reasoning about important mathematical ideas.

Way (2008) presents examples of good question types to stimulate mathematical thinking and recommends the use of Bloom’s Taxonomy (see Table 2).

<table>
<thead>
<tr>
<th>Cognitive process</th>
<th>What learners need to do</th>
<th>Action verbs for questioning</th>
</tr>
</thead>
<tbody>
<tr>
<td>Remember</td>
<td>Retrieve relevant information from long-term memory</td>
<td>Recognise, recall, define, describe, identify, list, match, reproduce, select, state, exemplify</td>
</tr>
<tr>
<td>Understand</td>
<td>Construct meaning from information and concepts</td>
<td>Paraphrase, interpret, classify, summarise, infer, compare, discuss, explain, rewrite</td>
</tr>
<tr>
<td>Apply</td>
<td>Carry out a procedure or use a technique in a given situation</td>
<td>Change, demonstrate, predict, relate, show how, solve, determine</td>
</tr>
<tr>
<td>Analyse</td>
<td>Separate information into parts and determine how the parts relate to one another</td>
<td>Analyse, compare, contrast, organise, distinguish, examine, illustrate, point out, relate, explain, differentiate, organise, attribute</td>
</tr>
<tr>
<td>Evaluate</td>
<td>Make judgements based on criteria and/or standards</td>
<td>Comment on, check, criticise, judge, critique, discriminate, justify, interpret, support</td>
</tr>
<tr>
<td>Create</td>
<td>Put elements together to form a coherent whole, or recognise elements into a new pattern</td>
<td>Combine, design, plan, rearrange, reconstruct, rewrite, generate, produce</td>
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The following set of questions represent examples for each of Bloom’s cognitive processes.

Remember: What shape is this?

Understand: How would you find the area of this shape?

Apply: Draw a shape like this one, measure its dimensions and calculate the area.

Analyse: Compare your shape with another student’s shape. How are they similar or different?

Evaluate: Choose another strategy to find the area of the shape you have drawn. Which approach is more efficient and why?

Create: Draw several more of these shapes with the same area. Which one has the largest perimeter? Comment on the change in perimeter with change in dimensions.

Task Types

There are many types of tasks, which could be used to allow students the opportunity to engage in problem solving and reasoning. Resources are available to support teachers with advice about types of tasks and good examples for use in primary, middle years or secondary classrooms. Some resources provide specific advice about problem solving (e.g., Booker & Bond, 2008) while others discuss strategies for enhancing reasoning amongst students in mathematics classrooms (e.g., McVarish, 2007).

The following sections present a few examples of open-ended questions, investigations, and games that lead to posing questions and exploring mathematical ideas. These task types encourage students to explore more challenging mathematics, to reflect on their understanding, to struggle with the process of expressing their ideas, and to write about their mathematical thinking. Providing opportunities for students to explore ideas in small groups and then to write about their learning supports the development of problem solving and reasoning. An important component of the process is to encourage careful recording of what the students do and what they learn. Much has been written about the purposes of using writing in mathematics classrooms including the recommendation that it can be used to assess students’ understandings as well as to promote learning. In addition, students’ writing exposes misconceptions and incomplete understandings.

Open-ended questions – When introducing open-ended questions and investigations into your classroom it should be remembered that students do not always find expressing or recording their ideas easy. We should not assume that students automatically write well; there is a need to spend time on this process.
To introduce open-ended questions in mathematics lessons:

- initially, be explicit about what is required, break down the question into manageable parts and request multiple solutions;
- prepare preliminary prompts for students who are having trouble getting started and extension questions for students who finish quickly;
- encourage students to ask themselves the following questions which involve the use of metacognitive strategies: what is the question asking?, what do I know about this?, how can I check this is correct?, have I recorded all that I know?, are there more answers that satisfy these conditions?, and can I summarise my ideas or make general statements?;
- conduct whole class discussion so that ideas and strategies can be shared and compared.

Further advice and many examples can be found in Sullivan and Lilburn (2004), Schuster and Anderson (2005) or Small (2009). The following examples of open-ended questions are grouped into the content strands.

**Number and Algebra:**

Find numbers between \( \frac{1}{8} \) and \( \frac{1}{5} \) and justify your choices.

A number was rounded to 2.15, what could the number have been?

**Chance and Data:**

If the average of a set of 5 scores was 12, what might the scores have been?

Design a 2 colour spinner where the probably of spinning a blue will be \( \frac{2}{3} \).

**Measurement and Geometry:**

Create several rectangles with an area of 24 cm\(^2\) and investigate their perimeters. What do you notice?

What are the possible dimensions of a container which could hold 1 litre of milk?

Investigations – The introduction of investigations requires careful planning and preparation. It is suggested that there should be preliminary discussion about how an investigation should be conducted and how it can then be presented in report format. Small investigations should precede the implementation of an individual extended task or project. Whole class discussion of investigations undertaken by small groups will help to guide future work and there needs to be feedback on writing attempts at this preliminary stage. If investigations are used for assessment purposes, students need to understand how important it is to show all of their thinking. It is helpful to discuss with the students what an exemplary solution might look like and what components are critical to the solution process. This will enable them to judge whether their own solution meets the desired criteria. Perhaps students could be given work samples and asked to discuss how well the question has been answered using an appropriate set of characteristics or criteria. Further advice about investigations is presented in Quinell (2010). Two other useful resources are Swan (2002) for good examples of investigations and Downton et al. (2006) for examples of rich assessment tasks with student work samples.

Examples of Investigations

1. The number 12 has six factors: 1, 2, 3, 4, 6 and 12.
   
   Four of these are even (2, 4, 6 and 12) and two are odd (1 and 30).
   
   Find some numbers which have all their factors, except 1, even.
   
   Describe the sequence of numbers that has this property.”

2. **Money Measurement** (Downton et al., 2007)

   You have won a prize. What would you choose – explain why?
   
   A. One square metre of 5 cent coins (the coins fill the square metre, and are lying flat, touching);
   B. A one litre milk carton filled with 20 cent coins;
   C. One Kilogram of $1 coins;
   D. A one metre long line of $2 coins (lying flat and touching).
3. **Map of Australia (Maths 300)**

If the area of the state of NSW represented 1 square unit, what would be the areas of each of the other states? Students estimate by visualizing, then discuss with a peer before revising their estimates. They further revise estimates after viewing a map of Australia. I then pose the question, ‘what strategy would you like to use now to get a more accurate estimate?’ to see what they suggest. In small groups, students are encouraged to collect data or apply their strategy to refine possible area comparisons.

4. **The Highway Problem** (Flewelling, 2005, p. 22)

A Grade 7 class discusses the loss of farmland to urban sprawl. The following questions arise:

How much land has to be taken out of production to build a new 36 kilometre-long, 2-lane highway?

Students discuss the problem in small groups, make (and test) assumptions, make (and refine) estimates, finally making estimates of the amount of land taken out of production.

A final recommendation for investigations is the wonderful resource developed by Phillips (2006) from the UK who has taken many photographs of everyday contexts to stimulate students posing mathematical questions.

Games – Games are excellent vehicles to practise number skills, to highlight misconceptions, and as a stimulus to investigate mathematical ideas. They are motivational and students enjoy the opportunity to practise work in a supportive classroom environment. Once students are aware of the rules, games can then be played regularly as part of lessons, or, as review and consolidation. Games allow students to actively participate in lessons and to practise basic skills in a non-threatening way. They encourage interaction and can be used by the teacher as a catalyst for mathematical discussion. Games need to be carefully chosen to support program objectives and can often be easily adapted to suit individual student needs. Close monitoring of student engagement is necessary to ensure the development of deep knowledge and skills is occurring.

Many games can be used to support the development of deep knowledge and skills of aspects of number. Teachers will be familiar with Bingo, Concentration and similar card matching activities such as adaptations of Dominoes, but there are many other games that can be played. Some games can be played initially as whole class activities but many games are best played between two players or in small groups. After playing a game several times, students should always be encouraged to talk about what they have learned, to record some of their observations and successful strategies, and to discuss what mathematics was needed to play the game. The following is an example of a game that promotes thinking about factors and multiples. For further games ideas see Gough (2010), Swan (2003), and Tickle and Burnett (2005).

**Multiples Bingo or Multo** (Downton et al., 2007) - this game provides opportunities for students to practise basic multiplication facts but it also allows for the investigation of number patterns and relationships. The game is best played as a whole class activity but it could then be played by small groups of students.

To play the game:

- write each of the multiplication facts from 1×1 to 10×10 on 100 separate cards (blank system cards are useful),
- ask students to draw a 4×4 grid and enter 16 different answers to the multiplication facts, for example

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<td>80</td>
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<td>35</td>
<td>8</td>
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<td>36</td>
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</table>

- shuffle the cards and read out the multiplication fact questions from the top of the deck,
- students cross out any correct answers they have recorded on their grid,
- students call out “bingo” when they have crossed out four in a row, four in a column, four in a diagonal, or the four corners,
- continue the game until each student has called out “bingo” on at least one occasion.

This game then leads to a discussion about the best 16 numbers to choose and what the ideal game board might look like.

Colour In Fractions - this game uses an area model to explore equivalent fractions. The game is best played in pairs. Use of this game to address a range of important ideas in fraction learning was recently reported in Clarke and Roche (2010).

To play the game:

• each player needs a copy of the game board below and each pair of players needs 2 dice;
• one die needs to have the numbers 1, 2, 2, 3, 3, and 4 recorded on the six faces. These represent the numerators of the fractions;
• the other die needs to have \( \frac{2}{3}, \frac{3}{4}, \frac{6}{8}, \frac{8}{12} \) recorded on each of the six faces;
• students take turns rolling the two dice together to make a fraction. This fraction, or an equivalent, is shaded on the student’s game board;
• the winner is the first to shade their own board completely.

This game can then be used as a stimulus to investigate which fractions are most likely to occur, as well as to explore the range of equivalent fractions.

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Final Advice for Teachers

To support students’ development of problem-solving competence, the problem-solving process has been presented as a series of stages or related processes (e.g., Polya, 1957; Garafalo & Lester, 1985). If working mathematically is described as a series of processes such as questioning, applying strategies, communicating, reasoning and reflecting, these terms can be linked to demonstrate a way teachers could use them to reflect the problem-solving process (see Table 3). However, representing these strategies as a list is somewhat misleading since solving problems is a messy process with problem solvers moving backwards and forwards between these stages as they test ideas, try a range of strategies, and grapple with making judgments about their solutions (Ben-Hur, 2006).

Table 3 – Stages of the problem-solving process

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<tr>
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</thead>
<tbody>
<tr>
<td>Understand the problem</td>
<td>Orientation</td>
<td>Questioning</td>
<td></td>
</tr>
<tr>
<td>Devise a plan</td>
<td>Organisation</td>
<td>Applying Strategies</td>
<td></td>
</tr>
<tr>
<td>Carry out the plan</td>
<td>Execution</td>
<td>Reasoning and communicating</td>
<td></td>
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<tr>
<td>Look back and examine the solution</td>
<td>Verification</td>
<td>Reflecting</td>
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Focusing on the stages of the problem solving process is just one aspect of developing problem-solving competence. Developing successful problem solvers is a complex task requiring a range of skills and dispositions (Stacey, 2005). Students need deep mathematical knowledge and general reasoning ability as well as heuristic strategies for solving non-routine problems. It is also necessary to have helpful beliefs and personal
attributes for organizing and directing their efforts. Coupled with this, students require good communication skills and the ability to work in cooperative groups (Figure 1). When planning to implement problem solving, it is important to consider all of these skills and dispositions.

![Diagram showing factors contributing to successful problem solving]

*Figure 1. Factors contributing to successful problem solving (Stacey, 2005, p. 342)*

It is an ongoing challenge for teachers to develop successful problem solvers given the constraints acting against teachers’ intentions and best efforts. The challenges for teachers are to:

- ask fewer questions in each lesson but use questions or problems which require reasoning;
- use rich problem-solving tasks including investigations and open-ended questions;
- discuss with students the role of problem solving in learning mathematics; and
- allocate time for students to grapple with the underlying mathematical ideas.

**Concluding Comments**

As noted in the *Shape of the Australian Curriculum: Mathematics* (NCB, 2009, p. 9), for many students, their “experience of mathematics is alienating and limited”, particularly given the frequency of use of low complexity problems (Stacey, 2003). If problem solving is to be promoted as an important component of the curriculum the types of problems which are most desirable must be made explicit.

Curriculum developers recognise that providing problem-solving experiences is critical if students are to be able to use and apply mathematical knowledge in meaningful ways. It is through problem solving that students develop deeper understanding of mathematical ideas, become more engaged and enthused in lessons, and appreciate the relevance and usefulness of mathematics.

Given the efforts to date by many countries (including Australia) to include problem solving as an integral component of the mathematics curriculum and the limited implementation in classrooms, it will take more than rhetoric to achieve this goal. While providing valuable resources and more time are important steps, it is possible that problem solving in the mathematics curriculum will only become valued when it is included in high-stakes assessment. In addition, teachers need readily available examples of useful non-routine problems, particularly in textbooks.

**References:**


Board of Studies NSW (2003). Mathematics Years 7-10 syllabus. Sydney: BOS NSW.